Agenda

• Comparing Numbers Learning Trajectory
• Symbolic Representations of Word Problem Situations
• Written Representations of Word Problem Situations
• Lunch
• Addition & Subtraction Learning Trajectory
• Equality
• Equality Learning Trajectory
• Individual Project Work Time
Symbolic Representations of Word Problems
Symbolic Representations

SITUATION EQUATIONS
• Reflect the action in the story in the manner in which happens

SOLUTION EQUATIONS
• Reflect the student’s thinking in the order or manner they solved the problem
What might an equation look like for each of these problem types?

Add to (Join) Result Unknown:

Connie had 5 marbles. Juan gave her 8 more marbles. How many marbles does Connie have altogether?

5 + 8 = _____ (situation equation)

5 + 8 = _____ (solution equation)
What might an equation look like for each of these problem types?

Take From (Separate) Result Unknown:

Connie had 13 marbles. She gave 5 to Juan. How many marbles does Connie have left?

13 – 5 = _____ (situation equation)
13 – 5 = _____ (solution equations)
What might an equation look like for each of these problem types?

Add to (Join) Change Unknown:

Connie has 5 marbles. How many more marbles does she need to have 13 marbles altogether?

\[ 5 + \_\_\_ = 13 \] (situation equation)

\[ \_\_\_ = 13 - 5 \] (possible solution equation)
What might an equation look like for each of these problem types?

Take From (Separate) Change Unknown:

Connie had 13 marbles. She gave some to Juan. Now she has 5 marbles left. How many marbles did she give to Juan?

13 - _____ = 5 (situation equation)
_____ = 13 – 5 (possible solution equation)
What might an equation look like for each of these problem types?

Take From (Separate) Start Unknown:

Connie had some marbles. She gave 5 to Juan. Now she has 8 marbles left. How many marbles did Connie have to start with?

_____ − 5 = 8 (situation equation)
_____ = 8 + 5 (possible solution equation)
What might an equation look like for each of these problem types?

Add to (Join) Start Unknown:

Connie had some marbles. Juan gave her 5 marbles. Now she has 13 marbles. How many marbles did she have to start with?

______ + 8 = 13 (situation equation)
8 + _____ = 13 (possible solution equations)
13 − 8 = 5
We are learning

• the CCSSM expectations around story problem structures.
• that students progress through levels of reasoning as they solve story problems.
• how we might support students’ representations of their thinking.
Representing Level 1 and Level 2 Thinking
“As children progress to Level 2 strategies they no longer need representations that show each quantity as a group of objects.”

-p. 16 OA Progressions

As children leave behind the need to represent each quantity, what implications does that have for us as teachers?
What’s missing from this sequence?
Margaret has 4 toy cars. Cassandra has 3 toy cars. How many toy cars do they have altogether?

Physical
Use concrete objects to form two groups and put the two groups together.

Symbolic
4 + 3 = 7
Representational Math Drawings

Drawing pictures that represent concrete objects provides a bridge to help young children connect their concrete representations to the abstract world of mathematical symbols.

“Math drawings facilitate reflection and discussion because they remain after the problem has been solved.”

Children need many opportunities to create such drawings.
What is a tape drawing?

A drawing that looks like a segment of tape, used to illustrate number relationships. Also known as strip diagrams, bar model, fraction strip or length model.

--CCSSM Glossary p. 87
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<td>Visual Model for Problem Solving (7RP1-3) Number Line Diagram (7.NS-1)</td>
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“These diagrams are a major step forward because the same diagrams can represent the adding and subtracting situations for all kinds of numbers students encounter in later grades (multi-digit whole numbers, fractions, decimals, variables).”

--p. 17 OA Progressions
Margaret has 4 toy cars. Cassandra has 3 toy cars. How many toy cars do they have altogether?

Physical

Use concrete objects to form two groups and put the two groups together.

Symbolic

4 + 3 = 7

1,2,3,4,5,6,7...7 cars
Margaret has 4 toy cars. Cassandra has 3 toy cars. How many toy cars do they have altogether?
Progression of Tape Diagrams

• Students begin by using concrete objects and drawing pictorial models

• Evolves into using bars to represent quantities
  – Enables students to become more comfortable using letter symbols to represent quantities later at the secondary level (Algebra)
Sara has 2 apples. Jon has 5 apples. How many apples do they have altogether? How many more apples does Jon have than Sara?
Tape Diagram Practice
Part-Part-Whole
Example A

Connor has 6 toy cars. He gets 8 more toy cars for his birthday. How many toy cars does he have now?

\[ 6 + 8 = 14 \]

He has 14 toy cars altogether.
Example B

Laura has a vase with 13 flowers. She puts 7 more flowers in the vase. How many flowers are in the vase?

There are 20 flowers in the vase.

13 + 7 = 20
Example C

174 children went to summer camp. If there were 93 boys, how many girls were there?

93 + ? = 174
174 – 93 = ?
There were 81 girls.
Example D

Julie brought 4 pieces of watermelon to a picnic. After Amy brings her some more pieces of watermelon, she has 9 pieces. How many pieces of watermelon did Amy bring Julie?

\[ 4 + ? = 9 \]
\[ 9 - 4 = ? \]

Amy brought 5 pieces of watermelon to Julie.
Example E

• Some yellow beads were on Sally’s bracelet. After she put 14 purple beads on the bracelet, there were 18 beads. How many yellow beads did Sally’s bracelet have at first?

$? + 14 = 18$
$18 - 14 = ?$
Sally had 14 yellow beads at first
Kiana found some shells at the beach. She gave 8 shells to her brother. Now she has 9 shells left. How many shells did Kiana find at the beach?
Stop to Reflect

What did you notice about the structure of the problems in Problem Set 1?

They were all addition or subtraction problems, and were all conducive to use of the Part-Whole model.
Part - Whole Model
Addition & Subtraction

Part + Part = Whole
Whole - Part = Part
Tape Diagram
Practice
Comparison
Example A

Danielle wrote 8 letters. Katrina wrote 12 letters. How many more letters did Katrina write than Danielle?

12 - 8 = 4

Katrina wrote 4 more letters than Danielle.
Nicole saw 14 foxes at the zoo. She saw 5 more monkeys than foxes. How many monkeys did Nicole see?

\[
14 + 5 = 19
\]

Nicole saw 19 monkeys at the zoo.
Stop to Reflect

What did you notice about the structure of problems in Problem Set 2?

They were all addition or subtraction comparison problems, and were all conducive to use of the Additive Comparison model.
The Comparison Model
Addition & Subtraction

larger quantity - smaller quantity = difference
smaller quantity + difference = larger quantity
Practice

• Return to the story problems you created.
• Create a tape diagram which illustrates each story you and your partner wrote.
Learning Intention & Success Criteria

We are learning

• the CCSSM expectations around story problem structures.
• that students progress through levels of reasoning as they solve story problems.
• how we might support students’ representations of their thinking.
Reflection/Summary

- Summarize some key points and classroom ideas related to the topics or focus standards in this session.

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<th>Focus Topics or Standards</th>
<th>Summary of Key Points</th>
<th>Classroom Ideas to Try</th>
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Learning Trajectory for Addition and Subtraction
Examining the Trajectory

• 1\textsuperscript{st} Pass – Look for Problem Types

• 2\textsuperscript{nd} Pass – Look for Levels of Thinking
Reflecting on the Trajectory

• What do you see that confirms what we’ve discussed the past two days?

• What surprises you?
Brain Break!

EGG
CHICKEN
DINOSAUR
HAWK!

Rock, Paper, Scissors, SHOOT!
Relational Thinking
Understanding Equality

Alyssa Murphy
Michelle Painter

Mequon-Thiensville School District
Agenda

- Warm up
- What does the equal sign mean?
- Student work/Levels of equality
- Ways to promote understanding of the equal sign
- Relational thinking practice
- Questions
Learning Intention and Success Criteria

Learning Intention: We are learning to examine the concept of equality.

Success Criteria: We will be successful when we can...
- understand what the equal sign means
- look at equations and solve them relationally
Solve the equations below:

$$48 + 24 = \underline{\hspace{2cm}} + 27$$

$$8 + 4 = \underline{\hspace{2cm}} + 5$$

Keep track of your strategy
-Think, then turn and talk.
-Share your strategy and thoughts with a colleague at your table.

\[ 48 + 24 = \underline{\quad} + 27 \]
\[ 8 + 4 = \underline{\quad} + 5 \]
Next discuss...

- How do you think your students would solve this problem?
- Would their strategy be different than yours?
- Why do you think they would answer that way?

As you talk through these questions, begin to think about a working definition of the equal sign that might make sense to both you and your students.
What does the equal sign mean?

The equal sign means, it can mean when two numbers
What is equality? Why is it important?

• Mathematical equality can be loosely defined as the principle that two sides of the equation have the same value and are thus interchangeable. (Kieran, 1981; Wittgenstein, 1961)

• Knowledge of the equal sign as an indicator of mathematical equality is foundational to children’s mathematical development and serves as a key link between arithmetic and algebra. (Matthews, 2012)
1.OA.7

Work with addition and subtraction equations.

Understand the meaning of the equal sign, and determine if equations involving addition and subtraction are true or false. *For example, which of the following equations are true and which are false*

- $6=6$
- $7=8-1$
- $5+2=2+5$
- $4+1=5+2$
Student Work

Take a look at the student work samples with your shoulder partner.

Group the samples into 2 groups:
  “got it” and “didn’t get it”

Guiding Questions...
  1. How did students arrive at their answers?
  2. What might this work show us about students’ understanding of the equal sign?
Matthews Levels of Understanding the Equal Sign

**Level One:** Rigid Operational

**Level Two:** Flexible Operational

**Level Three:** Basic Relational

**Level Four:** Comparative Relational
4th Grade Success: $48 + 24 = n + 27$
2nd Grade Success: 8 + 4 = ___ + 5
# Levels of Student Understanding of Equality

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<thead>
<tr>
<th>Level</th>
<th>Description</th>
<th>Core Equation Structures</th>
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| **Level 1. Rigid Operational** | Students can solve equations or evaluate true-false statements successfully that only have operations on the left side of the equal sign. | Equations with operations on left: \(a + b = c\)  
  \[4 + 8 = 7\] \(T\) or \(F: 3 + 4 = 7\)  
  \[3 + 4 = 8\] \(T\) or \(F: 3 + 4 = 8\)  
  \[5 + 4 = 8\] \(T\) or \(F: 5 + 4 = 8\) |
| **Level 2. Flexible Operational** | Students can successfully solve equations with operations on the right side of the equal sign or interpret statements that have no operations. | Equations with operations on right: \(c = a + b\)  
  \[5 = 3 + 4\] \(T\) or \(F: 8 = 3 + 4\)  
  \[7 = 5 + 4\] \(T\) or \(F: 7 = 3 + 4\)  
  Equations with no operations: \(a = a\)  
  \[7 = 7\] \(T\) or \(F: 7 = 7\)  
  \[8 = n\] \(T\) or \(F: n = n\) |
| **Level 3. Basic Relational** | Students can successfully solve or evaluate statements with operations on both sides of the equal sign, and explain or give correct relational definitions of the equal sign. | Equations with operations on both sides: \(a + b = c + d\)  
  \[a + b = c + d\] \(a + b - c = d + e\)  
  \[5 + 7 = 6 + 5\] \(6 \times 5 + 25 = 8 + 20\)  
  \[7m + 3m + 5 = m + 5\] \(-9\) |
| **Level 4. Comparative Relational** | Students can successfully use short-cuts (e.g., compensation strategies) and properties of the operations to solve equations or evaluate statements. | Equations that can be most efficiently solved by applying simplify transformations:  
  Use a shortcut to tell if the equation \("67 + 86 = 68 + 85"\) is true or false?  
  Figure out the value of \(m\) in \("3 \times 27 = 60 + m\) without fully multiplying out \(3 \times 27\). |

Student Work Sample Levels

Level 1: C, D, F, G, H, J

Level 2: none

Level 3: A, K (E)

Level 4: B, E, I, (E)
Ways to promote the understanding of =

**Lower grades**
- Representations (balances, ten frames, rekenreks, counters)
- True/False statements
- Word Problems

**Upper grades**
- True/False statements
- Solving for unknown variable
- Deep mathematical discourse
Questions
Disclaimer

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University of Wisconsin-Milwaukee, 2015-2018

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